the initial condition for which can be written in the form $\beta(0) = 0$. Solving this equation, we obtain the following expression for the function $f_1(t)$:

$$f_1(t) = \beta\left(\frac{t^2}{4}\right) = c_0\left[1 - \exp\left(-\frac{3t^{4/3}}{8\eta}\right)\right]$$

and from this it follows that

$$\lim_{z\to+0} c(\xi, z) = \frac{c_0}{\Gamma(2/3)}\int_0^\infty \exp\left[-\frac{3}{2\eta}(\xi\rho)^{2/3}\right] e^{-\rho}\rho^{-1/3}\,d\rho \qquad (A6)$$

The result (A6) obtained together with the first boundary condition of (A3), enables us to determine the diffusive flux on the surface of the sphere which is equal, with accordance with the second boundary condition of (2), to $k \lim_{z\to+0} c(\xi, z)$, and this leads to formula (5).

The author thanks Iu. P. Gupalo and Iu. S. Riazantsev for unceasing interest and valuable discussions.

REFERENCES

1. Gupalo, Iu. P. and Riazantsev, Iu. S., On mass and heat transfer from a spherical particle in a laminar stream of viscous fluid. PMM Vol. 35, № 2, 1971.
2. Levich, V. G., Physico-Chemical Hydrodynamics. (English translation), Prentice-Hall, Englewood Cliffs, New Jersey, 1962.
3. Sutton, W. G. L., On the equation of diffusion in turbulent medium. Proc. Roy. Soc., A, Vol. 182, p. 43, 1943.

Translated by L. K.

UDC 539. 3: 534. 1

DIFFRACTION OF A SPHERICAL ELASTIC WAVE BY A WEDGE

PMM Vol. 40, No. 5, 1976, pp. 898 - 908

V. B. PORUCHIKOV
(Moscow)

(Received February 18, 1976)

A three-dimensional nonstationary problem of spherical elastic wave diffraction by a smooth solid wedge with arbitrary apex angle is considered. An exact solution in the form of a sum of two terms, the known acoustic solution and an additional part describing the influence of elasticity, and caused by the appearance of additional longitudinal and transverse diffraction waves, is obtained by the method of integral transforms with extraction of the singularities in the neighborhood of an edge. This latter term essentially distinguishes the elastic from the acoustic solution. The particular case of an incident wave with a jump in the stresses at the front is investigated in detail.

The corresponding acoustic problem has been examined in [1-4], where the solution in elementary functions was first obtained in [2]. Only the solution for the plane wave diffraction problem [5] is known for a wedge in the elastic case. Solutions found earlier for plane diffraction problems by a smooth wedge [6] and a smooth half-plane [7], which agree with the solution of the corresponding acoustic problems, are not true because of neglecting the condition at the edge, which indeed resulted in nonintegrable stresses in the neighborhood of the edge.

1. Let us consider an elastic medium with the propagation velocities a and b for longitudinal and transverse waves filling a domain $r > 0,\ 0 < \theta < \pi / l$, $-\infty < z < \infty$ and bounding the wedge $(\pi / l < \theta < 2\pi)$ on whose side walls $\theta = 0,\ \theta = \pi / l$ are given the conditions: $v_\theta = 0,\ \sigma_{\theta r} = \sigma_{\theta z} = 0$, where $r,\ \theta,\ z$ are cylindrical coordinates (the z-axis coincides with edge of the wedge), v_θ is the displacement vector component and $\sigma_{\theta r},\ \sigma_{\theta z}$ are stress tensor components. At the instant $\tau = -r_0\ (\tau = at,\ t$ is the time, $r_0 > 0)$, a source of a spherical elastic longitudinal wave with the potential [8]

$$\varphi_0 = f(\tau + r_0 - R) / R,\quad R = [z^2 + r^2 + r_0^2 - 2rr_0 \cos\ (\theta - \theta_0)]^{1/2} \tag{1.1}$$

whose front reaches the wedge surface at the instant $\tau_*\ (-r_0 < \tau_* \leqslant 0)$ starts to act at the point $(r_0,\ \theta_0,\ 0)$, where $f(\tau)$ is an arbitrary function satisfying the conditions of applying the Laplace transform and $f(\tau) \equiv 0$ at $\tau < 0$. The wedge introduces the perturbation $\mathbf{u} \equiv \{u_r,\ u_\theta,\ u_z\}$, described by a longitudinal φ and two scalar transverse $\psi_1,\ \psi_2$ potentials by means of the formulas [8]

$$u_r = \frac{\partial\varphi}{\partial r} + \frac{\partial\psi_1}{r\partial\theta} + \frac{\partial^2\psi_2}{\partial r\partial z},\quad u_\theta = \frac{\partial\varphi}{r\partial\theta} - \frac{\partial\psi_1}{\partial r} + \frac{\partial^2\psi_2}{r\partial\theta\partial z} \tag{1.2}$$
$$u_z = \frac{\partial\varphi}{\partial z} - \frac{\partial^2\psi_2}{r^2\partial\theta^2} - \frac{\partial}{r\partial r}\left(r\frac{\partial\psi_2}{\partial r}\right)$$

into the field of incident wave displacement.

As can be confirmed, the boundary conditions on the wedge will hence be satisfied if compliance with the conditions $\partial\varphi / \partial\theta = -\partial\varphi_0 / \partial\theta,\ \ \psi_1 = 0$ and $\partial\psi_2 / \partial\theta = 0$ at $\theta = 0,\ \pi / l$ is required. Consequently, taking into account that perturbations do not occur prior to the instant $\tau = \tau_*$, we obtain the following three systems of equations, boundary and initial conditions to determine φ, ψ_1 and ψ_2:

$$\Delta\varphi = \partial^2\varphi / \partial\tau^2 \quad (\Delta \equiv \partial^2 / \partial r^2 + r^{-1}\partial / \partial r + r^{-2}\partial^2 / \partial\theta^2 + \partial^2 / \partial z^2) \tag{1.3}$$
$$\partial\varphi / \partial\theta = -\partial\varphi_0 / \partial\theta \quad \text{for} \quad \theta = 0,\ \ \pi / l,\quad \varphi = 0 \quad \text{at} \quad \tau < \tau_*$$

$$\Delta\psi_j = \gamma^2\partial^2\psi_j / \partial\tau^2 \quad (\gamma = a / b > 1,\ j = 1,\ 2) \tag{1.4}$$

$$\psi_1 = \partial\psi_2 / \partial\theta = 0 \quad \text{for} \quad \theta = 0,\ \ \pi / l,\quad \psi_j = 0 \quad \text{at} \quad \tau < \tau_*$$

These three systems are mutually connected by the following condition *) on the wedge edge:

$$\mathbf{u} = \text{const} + O(r^\varepsilon),\quad \varepsilon > 0 \quad \text{for} \quad r \to 0 \tag{1.5}$$

which assures integrability of the stresses (as $r \to 0$) and uniqueness of the solution of the problem formulated. It is assumed that condition (1.5) is satisfied uniformly in $\tau,\ \theta,\ z$.

*) See Kostrov, B. V., Some Dynamical Problems of Mathematical Elasticity Theory. Kandidat Dissertation, Moscow State University, 1964.

Moreover, in solving the problem we consider that $1/2 \leqslant l < 1$ since the solution for $l \geqslant 1$ can be obtained from the symmetric part (relative to the bisector plane of the wedge) of the solution found for $1/2 \leqslant l < 1$.

2. Applying a two-sided Laplace transform in τ and z successively to (1.3) and (1.4), and then expanding the transforms of the required potentials $\bar{\varphi}^*(p, r, \theta, s)$ and $\bar{\psi}_2^*(p, r, \theta, s)$ in a cosine series, and $\bar{\psi}_1^*(p, r, \theta, s)$ in a sine series in the segment $[0, \pi / l]$, we obtain the following second-order ordinary differential equations to determine the coefficients of these expansions:

$$\frac{d^2a_n}{dr^2} + \frac{da_n}{rdr} - \frac{n^2l^2a_n}{r^2} = \omega^2 a_n + f_n(r) \tag{2.1}$$

$$\frac{d^2b_{nj}}{dr^2} + \frac{db_{nj}}{rdr} - \frac{n^2l^2}{r^2} b_{nj} = \varkappa^2 b_{nj} \qquad (j = 1, 2) \tag{2.2}$$

$$\bar{\varphi}^*(p,r,\theta,s) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nl\theta, \quad a_n = 2\pi^{-1}l \int_0^{\pi/l} \bar{\varphi}^* \cos nl\theta d\theta \qquad (n = 0, 1, 2, \ldots)$$

$$\bar{\psi}_1^*(p,r,\theta,s) = \sum_{n=1}^{\infty} b_{n1} \sin nl\theta, \quad b_{n1} = 2\pi^{-1}l \int_0^{\pi/l} \bar{\psi}_1^* \sin nl\theta d\theta \qquad (n = 1, 2, 3, \ldots)$$

$$\bar{\psi}_2^*(p,r,\theta,s) = \frac{b_{02}}{2} + \sum_{n=1}^{\infty} b_{n2} \cos nl\theta, \quad b_{n2} = 2\pi^{-1}l \int_0^{\pi/l} \bar{\psi}_2^* \cos nl\theta d\theta \qquad (n = 0, 1, 2, \ldots)$$

$$\bar{\varphi}^*(p,r,\theta,s) = (2\pi i)^{-1} \int_{-\infty}^{\infty} \bar{\varphi}(p,r,\theta,z)\, e^{-sz} dz$$

$$\bar{\varphi}(p,r,\theta,z) = (2\pi i)^{-1} \int_{-\infty}^{\infty} \varphi(\tau,r,\theta,z)\, e^{-p\tau} d\tau$$

$$\bar{\psi}_j^*(p,r,\theta,s) = (2\pi i)^{-1} \int_{-\infty}^{\infty} \bar{\psi}_j(p,r,\theta,z)\, e^{-sz} dz$$

$$\bar{\psi}_j(p,r,\theta,z) = (2\pi i)^{-1} \int_{-\infty}^{\infty} \psi_j(\tau,r,\theta,z)\, e^{-p\tau} d\tau$$

$$f_n(r) = \frac{2l}{\pi r^2} \left[(-1)^n \frac{\partial \bar{\varphi}_0^*}{\partial \theta}\bigg|_{\theta=\pi/l} - \frac{\partial \bar{\varphi}_0^*}{\partial \theta}\bigg|_{\theta=0} \right], \quad \operatorname{Re} p > 0$$

$$|\operatorname{Re} s| < \operatorname{Re} p, \quad \omega = (p^2 - s^2)^{1/2}, \quad \varkappa = (\gamma^2 p^2 - s^2)^{1/2}$$

$$\bar{\varphi}_0^*(p,r,\theta,s) = \bar{f}(p)\, e^{pr_0} \int_{-\infty}^{\infty} e^{-sz-pR} R^{-1} dz = 2\bar{f}(p)\, e^{pr_0} K_0(\rho\omega)$$

$$f(\tau) \rightleftharpoons \bar{f}(p), \quad \rho = [r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)]^{1/2}$$

Here $K_\alpha(s)$ is the Macdonald function of order α of the argument s. (The inequality $\operatorname{Re} p > 0$ follows from the fact that integration with respect to τ in the two-sided Laplace transform occurs for $\tau > -r_0$ since the source does not act for $\tau < -r_0$). Slits to separate the branches of the functions ω and $\varkappa$ in the s plane are made from the points $s = \pm p$ (from the points $s = \pm\gamma p$) for $\varkappa$) to infinity along the rays $\arg s = \arg p$ and $\arg s = \pi + \arg p$, and the ω and $\varkappa$ branches are selected so that $\omega = p$ and $\varkappa = \gamma p$ for $s = 0$. Then it is seen that $\operatorname{Re} \omega > 0$, $\operatorname{Re} \varkappa > 0$ for $|\operatorname{Re} s| < \operatorname{Re} p$. Solving (2. 1) and (2. 2) we obtain

$$a_n = A_n K_{nl}(r\omega) + B_n I_{nl}(r\omega) + F_n(r) \tag{2.3}$$

$$F_n(r) = -K_{nl}(r\omega)\int_0^r I_{nl}(x\omega) f_n(x)\,x dx - I_{nl}(r\omega)\int_r^\infty K_{nl}(x\omega) f_n(x) x dx$$

$$b_{nj} = C_{nj}K_{nl}(r\varkappa) + D_{nj}I_{nl}(r\varkappa) \quad (j = 1, 2) \tag{2.4}$$

Here $I_\alpha(s)$ is a modified Bessel function of the first kind of order α. Using asymptotic expressions for the cylinder functions

$$K_\alpha(s) \sim [\pi / (2s)]^{1/2} e^{-s}, \quad I_\alpha(s) \sim (2\pi s)^{-1/2} e^{s}, \quad |s| \to \infty,$$

$$|\arg s| < \pi / 2$$

we find that $F_n(r) \to 0$ as $r \to \infty$. Since the perturbations are propagated at a finite velocity, we then assume that $|\bar\varphi| < CR^{-1}|e^{-pR}|$, $|\bar\psi_j| < CR^{-1}|e^{-pR}|$ as $R \to \infty$, where C is independent of r, θ, z. We hence obtain that $a_n \to 0$, $b_{nj} \to 0$ as $r \to \infty$. We then find from (2. 3) and (2. 4) $B_n \equiv D_{nj} \equiv 0$.
To determine the remaining coefficients A_n and C_{nj} , let us use condition (1. 5) by assuming that it is conserved even for the coefficients $\bar u_{rn}^*$ and $\bar u_{zn}^*$ of the cosine expansions of the transforms $\bar u_r^*$ and $\bar u_z^*$ and the coefficients $\bar u_{\theta n}^*$ of the sine series expansion of the transform $\bar u_\theta^*$:

$$\bar u_{rn}^* = \text{const} + O(r^\varepsilon), \quad \bar u_{zn}^* = \text{const} + O(r^\varepsilon), \quad \bar u_{\theta n}^* = \text{const} + O(r^\varepsilon)$$

$$\bar u_{rn}^* = 2l\pi^{-1}\int_0^{\pi/l} \bar u_r^* \cos nl\theta d\theta, \quad \bar u_{zn}^* = 2l\pi^{-1}\int_0^{\pi/l} \bar u_z^* \cos nl\theta d\theta \tag{2.5}$$

$$(n = 0, 1, 2, \ldots)$$

$$\bar u_{\theta n}^* = 2l\pi^{-1}\int_0^{\pi/l} \bar u_\theta^* \sin nl\theta\, d\theta \quad (n = 1, 2, 3, \ldots)$$

$$\bar u_k^* = \int_{-\infty}^{\infty} e^{-sz} dz \int_{-\infty}^{\infty} u_k e^{-p\tau} d\tau \quad (k = r, \theta, z)$$

$$(\varepsilon > 0,\ r \to 0)$$

Using the expression (1. 2) in the estimates (2. 5), we obtain a system of three equations for each n ($n = 0, 1, 2, \ldots$) as $r \to 0$

$$da_n / dr + r^{-1} nl b_{n1} + s db_{n2} / dr = \text{const} + O(r^\varepsilon) \tag{2.6}$$

$$s a_n - \varkappa^2 b_{n2} = \text{const} + O(r^\varepsilon) \quad (\varepsilon > 0)$$

$$-nl r^{-1} a_n - db_{n1} / dr - r^{-1} s b_{n2} nl = \text{const} + O(r^\varepsilon)$$

from which the coefficients A_n and C_{nj} in (2.3) and (2.4) for a_n and b_{nj} should be found (for $n = 0$ the system of three equations degenerates into a system of two equations for a_0 and b_{02} because $b_{01} = 0$). To determine A_n and C_{nj} we use the asymptotic expressions for the cylinder functions $I_\alpha(s)$ and $K_\alpha(s)$ as $s \to 0$

$$I_\alpha(s) = (s/2)^\alpha / \Gamma(1+\alpha) + O(s^{2+\alpha}) \tag{2.7}$$
$$K_0(s) = -\ln s + O(1), \quad K_1(s) = s^{-1} + O(s \ln s)$$
$$2K_\alpha(s) = \Gamma(\alpha)(2/s)^\alpha + \begin{cases} \Gamma(-\alpha)(s/2)^\alpha + O(s^{2-\alpha}), & 0<\alpha<1 \\ O(s^{2-\alpha}), & \alpha>1 \end{cases}$$

It can be shown by using (2.7) that the following asymptotic estimates for $F_n(r)$ are satisfied as $r \to 0$:

$$F_0(r) = \text{const} + O(r), \quad F_1(r) = Mr^l + O(r) \tag{2.8}$$
$$M = -[(\omega/2)^l / \Gamma(1+l)] \int_0^\infty K_l(x\omega) f_1(x) x\, dx =$$
$$8\bar{f}(p)[e^{pr_0}/\Gamma(l)] K_l(r_0\omega)(\omega/2)^l \cos l\theta_0$$
$$F_n(r) = O(r), \quad n \geqslant 2$$

(in particular, we have $F_{2n}(r) \equiv 0$ as $l = 1/2$ since $f_{2n}(r) \equiv 0$) in this case).

Substituting (2.3) and (2.4) into (2.6) and using the asymptotic estimates (2.7) and (2.8), it can be noted that conditions (2.6) will be satisfied for $n = 0$ and $n \geqslant 2$ if we set $A_n \equiv C_{nj} \equiv 0$. In the case $n = 1$ we obtain the following system from (2.6) :

$$Sr^{-l-1} + Tr^{l-1} + O(1) = \text{const} + O(r^\varepsilon), \quad Wr^{-l} + O(r^l) = \text{const} + O(r^\varepsilon)$$
$$Sr^{-l-1} - Tr^{l-1} + O(1) = \text{const} + O(r^\varepsilon) \quad (\varepsilon > 0) \tag{2.9}$$
$$S = -2^{l-1}\Gamma(1+l)[A_1\omega^{-l} - C_{11}\varkappa^{-l} + sC_{12}\varkappa^{-l}]$$
$$T = Ml - 2^{-l-1}\Gamma(1-l)[A_1\omega^l + C_{11}\varkappa^l + s\varkappa^l C_{12}]$$
$$W = 2^{l-1}\Gamma(l)[A_1 s\omega^{-l} - \varkappa^{2-l} C_{12}]$$

We find $S = 0, T = 0, W = 0$ from (2.9), which yields the following expressions for A_1, C_{11} and C_{12}:

$$A_1 = \omega^l \varkappa^{2-l} s^{-1} C_{12}, \quad C_{11} = \gamma^2 p^2 s^{-1} C_{12} \tag{2.10}$$
$$C_{12} = \frac{16}{\pi} \frac{l\bar{f}(p) s\omega^l K_l(r_0\omega) \sin l\pi \cos l\theta_0}{\omega^{2l}\varkappa^{2-l} + (s^2 + \gamma^2 p^2)\varkappa^l} e^{pr_0}$$

Consequently, by using (2.10) we obtain for $\bar{\varphi}^*$, $\bar{\psi}_1^*$ and $\bar{\psi}_2^*$

$$\bar{\varphi}^* = \frac{F_0(r)}{2} + \sum_{n=1}^\infty F_n(r) \cos nl\theta + s^{-1}\varkappa^{2-l}\omega^l \cos l\theta K_l(r\omega) C_{12} \tag{2.11}$$
$$\bar{\psi}_1^* = s^{-1}\gamma^2 p^2 \sin l\theta K_l(r\varkappa) C_{12}, \quad \bar{\psi}_2^* = \cos l\theta K_l(r\varkappa) C_{12}$$

The sum of the terms with F_n $(n = 0, 1, 2, \ldots)$ in the expression for $\overline{\varphi}^*$ from (2. 11) is the Laplace transform (in τ and z) for the perturbed solution of the corresponding acoustic problem φ_1. This is quite simple to prove if it is noted that the original of this sum firstly satisfies the system (1. 3), and secondly, it can be shown that this original satisfies a condition assuring the uniqueness of the solution of the acoustic problem (for sufficiently smooth functions $f(\tau)$) [9]

$$\partial (\varphi_0 + \varphi_1) / \partial \tau = O(1), \quad r\partial (\varphi_0 + \varphi_1) / \partial r = o(1), \quad r \to 0$$

Thus, by adding the transform of the incident spherical wave, and applying the inverse Laplace transforms in s and p, we obtain $(\Phi = \varphi_0 + \varphi)$

$$\Phi = \varphi_a + \frac{\cos l\theta}{(2\pi i)^2} \int\limits_{c_0 - i\infty}^{c_0 + i\infty} e^{p\tau}\, dp \int\limits_{b_0 - i\infty}^{b_0 + i\infty} s^{-1} \varkappa^{2-l} \omega^l C_{12} K_l(r\omega)\, e^{sz}\, ds. \tag{2.12}$$

$$\psi_1 = \gamma^2 \frac{\sin l\theta}{(2\pi i)^2} \int\limits_{c_0 - i\infty}^{c_0 + i\infty} e^{p\tau} p^2\, dp \int\limits_{b_0 - i\infty}^{b_0 + i\infty} s^{-1} C_{12} K_l(r\varkappa)\, e^{sz}\, ds$$

$$\psi_2 = \frac{\cos l\theta}{(2\pi i)^2} \int\limits_{c_0 - i\infty}^{c_0 + i\infty} e^{p\tau}\, dp \int\limits_{b_0 - i\infty}^{b_0 + i\infty} C_{12} K_l(r\varkappa)\, e^{sz}\, ds \quad (c_0 > |b_0|)$$

By using the results in [3], the acoustic solution $\varphi_a = \varphi_0 + \varphi_1$ can here be represented as

$$\varphi_a = \frac{f(+0)}{1 + \tau / r_0} Q\left(\tau + \frac{\tau^2 - r^2 - z^2}{2r_0}, r, \theta\right) - \int\limits_{R - r_0}^{\tau} \frac{d}{dx}\left[\frac{f(\tau - x)}{1 + x / r_0}\right] Q\left(x + \frac{x^2 - r^2 - z^2}{2r_0}, r, \theta\right) dx \tag{2.13}$$

where $Q(\tau, r, \theta)$ in (2. 13) is the solution of the acoustic problem of diffraction of a plane wave $\eta[\tau + r\cos(\theta - \theta_0)] / r_0$ by the wedge under consideration, which has the following form according to [9]:

$$Q(\tau, r, \theta) = \eta(r - \tau)\{\sigma(\theta - \theta_0)\eta[\tau + r\cos(\theta - \theta_0)] + \sigma(\theta + \theta_0)\eta[\tau + r\cos(\theta + \theta_0)^*]\} r_0^{-1} + \eta(\tau - r)\pi^{-1} r_0^{-1} (\operatorname{arctg} \lambda_+ + \operatorname{arctg} \lambda_-)$$

$$\lambda_\pm = \frac{(1 - y^{2l}) \sin l\pi}{(1 + y^{2l}) \cos l\pi - 2y^l \cos l(\theta \pm \theta_0)}, \quad y = \frac{\tau}{r} - \left[\left(\frac{\tau}{r}\right)^2 - 1\right]^{1/2}$$

$$\eta(x) = 1, \quad x > 0; \quad \eta(x) = 0, \quad x < 0$$

$$\sigma(\theta) = 1, \quad |\theta| < \pi; \quad \sigma(\theta) = 0, \quad \pi < |\theta| < \pi / l$$

$$\sigma(\theta + 2\pi / l) = \sigma(\theta), \quad (\theta + \theta_0)^* = \theta + \theta_0 + 2\pi m / l$$

Here the values of the arctangents are taken in the interval $(0, \pi)$, and the integer m $(m = 0, -1)$ is selected so that the inequality $-\pi / l < (\theta + \theta_0)^* \leqslant \pi / l$ will always be satisfied at the point of physical space under consideration.

As follows from (2. 12), the elastic terms supplementing the acoustic solution drop out only for $\theta_0 = \pi / (2l)$ (case of symmetry relative to the bisector plane of the wedge) and for $l \to 1$ (wave reflection from a plane wall). By using (2. 12) it can be shown that the displacements are bounded and the stresses are integrable in the neighborhood of the wedge edge.

By using the change of variable $q = s / p$ the expressions (2. 12) are represented as (if we set $b_0 = 0$)

$$\Phi = \varphi_a + \frac{\cos l\theta}{(2\pi i)^2} \int_{c_0-i\infty}^{c_0+i\infty} \bar{f}(p) e^{p(r_0+\tau)} p\, dp \int_L \xi(p, q)\, dq \tag{2.14}$$

$$\psi_1 = -\frac{\gamma^2 \sin l\theta}{(2\pi i)^2} \int_{c_0-i\infty}^{c_0+i\infty} \bar{f}(p) e^{p(r_0+\tau)} p\, dp \int_L \zeta(p, q)\, dq$$

$$\psi_2 = \frac{\cos l\theta}{(2\pi i)^2} \int_{c_0-i\infty}^{c_0+i\infty} \bar{f}(p) e^{p(r_0+\tau)} dp \int_L \zeta(p, q)\, q\, dq$$

$$\xi(p, q) = K_l(pr\sqrt{1-q^2})(1-q^2)^{l/2} \Phi_0 e^{pqz},$$

$$\zeta(p, q) = K_l(pr\sqrt{\gamma^2-q^2})(\gamma^2-q^2)^{l/2-1} \Phi_0 e^{pqz}$$

$$\Phi_0 \equiv \Phi_0(p, q) = \frac{16}{\pi} l K_l(pr_0\sqrt{1-q^2}) \frac{(1-q^2)^{l/2} \sin l\pi \cos l\theta_0}{(1-q^2)^l + (q^2+\gamma^2)(\gamma^2-q^2)^{l-1}}$$

The contour L is shown in Fig. 1, where $\alpha_0 = \pi / 2 - \arg p$ and the slits from the s plane go over into slits along the real axis between the points $\pm\gamma$ and ± 1 to infinity in the q plane (and $(\gamma^2 - q^2)^{1/2} = \gamma$, $(1 - q^2)^{1/2} = 1$ for $q = 0$).

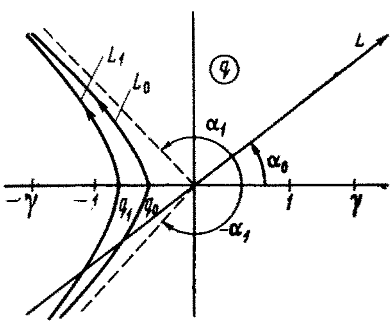

Fig. 1

It is sufficient to consider (2. 14) for $z \geqslant 0$ since Φ and ψ_1 are even while ψ_2 is an odd function of z (this is proved easily by using (2. 14)). Then it can be shown for $z > 0$ that the contour L in the expression for Φ can be deformed into the curve L_0, and the contour L in the expressions for ψ_1 and ψ_2 into the curve L_1. Points on these curves satisfy the equations

$$L_0:\ \operatorname{Im}[qz - (r + r_0)(1 - q^2)^{1/2}] = 0$$

$$L_1:\ \operatorname{Im}[qz - r(\gamma^2 - q^2)^{1/2} - r_0(1 - q^2)^{1/2}] = 0.$$

Both curves, whose shape in the q plane is shown in Fig. 1, are symmetric relative to the real axis and have the same asymptotes forming the angles $\pm\alpha_1$ with the real axis : $\operatorname{tg}\alpha_1 = -(r + r_0) / z$ upon removal from the origin. The points of intersection q_0 and q_1 of L_0 and L_1 with the real axis are determined, respectively, from the equations

$$z + (r + r_0)q(1 - q^2)^{-1/2} = 0,$$

$$z + rq(\gamma^2 - q^2)^{-1/2} + r_0 q(1 - q^2)^{-1/2} = 0$$

The function $N_0\ (q,\ z,\ r) \equiv qz - (r + r_0)(1 - q^2)^{1/2}$ is real on the curve L_0 and takes its maximum value on L_0 at the point q_0: $N_0\ (q_0,\ z,\ r) = - [z^2 + (r + r_0)^2]^{1/2}$.

Similarly, the function $N_1\ (q,\ z,\ r) \equiv qz - r\ (\gamma^2 - q^2)^{1/2} - r_0\ (1 - q^2)^{1/2}$, is real on L_1 and takes its maximum value on L_1 at the point q_1: $N_1\ (q_1,\ z,\ r) = -R_1\ (R_1 \equiv R_1\ (z,\ r))$. As $z \to 0$ the values of q_0 and q_1 tend to zero, and the curves L_0 and L_1 go over into the imaginary axis in the limit. Consequently, (2.14) are represented for $z \geqslant 0$ as

$$\Phi = \varphi_a + \cos l\theta \int_{L_0} \Phi_1\,(1 - q^2)^{l/2-1/4}\, dq \int_{-0}^{x_0} f'(x)\, U(x,\ q)\, dx \tag{2.15}$$

$$\psi_1 = \gamma^2 \sin l\theta \int_{L_1} \Phi_1\,(\gamma^2 - q^2)^{l/2-5/4}\, dq \int_{-0}^{x_1} f'(x)\, V(x,\ q)\, dx$$

$$\psi_2 = \cos l\theta \int_{L_1} \Phi_1\,(\gamma^2 - q^2)^{l/2-5/4}\, q\, dq \int_{0}^{x_1} f(x)\, V(x,\ q)\, dx$$

$$U(x,\ q) = P_{l-1/2}\left[1 + \frac{(x_0 - x)^2 + 2\,(x_0 - x)\,(r + r_0)\sqrt{1 - q^2}}{2rr_0\,(1 - q^2)}\right]$$

$$V(x,\ q) = P_{l-1/2}\left[1 + \frac{(x_1 - x)^2 + 2\,(x_1 - x)\,(r\sqrt{\gamma^2 - q^2} + r_0\sqrt{1 - q^2})}{2rr_0\sqrt{(\gamma^2 - q^2)\,(1 - q^2)}}\right]$$

$$x_0 = \tau + r_0 + qz - (r + r_0)\sqrt{1 - q^2}$$

$$x_1 = \tau + r_0 + qz - r\sqrt{\gamma^2 - q^2} - r_0\sqrt{1 - q^2}$$

Here $P_{l-1/2}(x)$ is the Legendre function of the first kind and the derivative $f'(x)$ in the expressions Φ and ψ_1 is understood in the generalized sense.

The operational calculus formula for the transform $Kl\,(ps)\,Kl\,(pq)\,e^{p(s+q)}$, which is written with an error in both the tables [10] (formula 60, section 16, chapter 5) and the handbook [11] where the Laplace-Carson transform (formula 29. 205) is given, was used to obtain (2. 15). To obtain its correct expression, let us represent this transform as

$$\frac{\pi}{2 \sin l\pi}\,[K_{-l}\,(ps)\,I_{-l}\,(pq) - K_l\,(ps)\,I_l\,(pq)]\,e^{p(s+q)}$$

by using the property $K_{-l}\,(s) = K_l\,(s)$ and $K_l\,(s) = \pi\,[I_{-l}\,(s) - I_l\,(s)]\,/\,(2 \sin l\,\pi)$. Then by using the operational calculas formula from [12] we obtain

$$K_l\,(ps)\,K_l\,(pq)\,e^{p(s+q)} \doteq \frac{\pi}{2\sqrt{sq}}\, P_{l-1/2}\left[\frac{(\tau + 2s)\,(\tau + 2q)}{2sq} - 1\right]\eta(\tau)$$

$$|\arg s| < \pi, \quad |\arg q| < \pi$$

(l is any complex number)

For $\max x_0\,(q) < 0$ ($\max x_0\,(q) = x_0\,(q_0) = \tau + r_0 - [z^2 + (r + r_0)^2]^{1/2}$) we find $\Phi = \varphi_a$ from (2. 15) and we have $\psi_1 = \psi_2 = 0$ for $\max x_1\,(q) < 0$ ($\max x_1\,(q) = x_1\,(q_1) = \tau + r_0 - R_1$) . Therefore, $\tau + r_0 = -[z^2 + (r + r_0)^2]^{1/2}$ and $\tau + r_0 = R_1$ are, respectively, the equations of the longitudinal and transverse diffraction wave fronts. The maps of the perturbed domains in the section $z =$ const $(\tau > (z^2 + r_0^2)^{1/2} - r_0)$ with and without shade are shown in Figs. 2a and 2b, respectively, where

$\angle BOA = \pi - \theta_0, \angle COA = 2\pi / l - \pi - \theta_0, \; < GOA = \pi + \theta_0$
(the angles are measured counter-clockwise from the ray OA), and the wavefronts 1 -5 are given respeetively by the equations :

$$1)\ \tau + r_0 = R\ (\theta),\ 2)\ \tau + r_0 = R\ (-\theta),\ 3)\ \tau + r_0 = [z^2 + (r + r_0)^2]^{1/2},\ 4)\ \tau + r_0 = R_1,\ 5)\ \tau + r_0 = R\ (2\pi / l - \theta),\ R\ (\theta) \equiv R = [z^2 + r^2 + r_0^2 - 2\, rr_0 \cos\ (\theta - \theta_0)]^{1/2}$$

(the coordinate θ is measured counter-clockwise from the ray OA , $\beta = 2\pi - \pi / l$).

The longitudinal and transverse diffraction wave fronts are described by the equations $r = \tau, r = \tau / \gamma$ in the $z = 0$ plane.

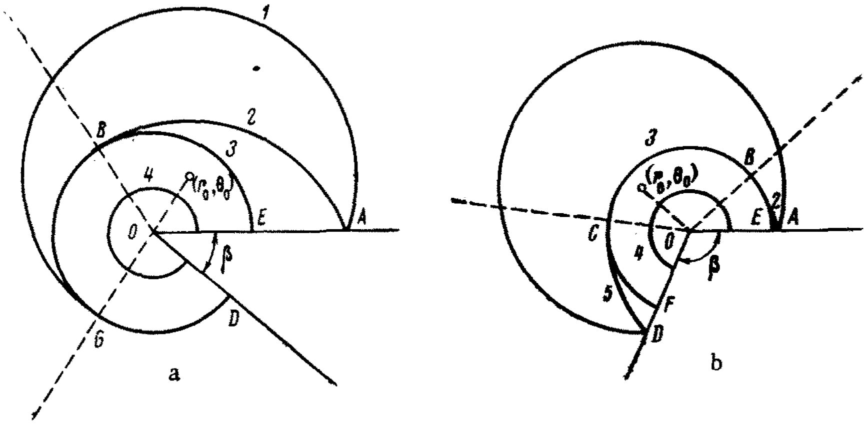

Fig. 2

It should be noted that taking account of the condition on the edge results pri - marily in a qualitative distinction between the solution of the elastic problem and the solution of the corresponding acoustic problem since transverse diffraction waves appear in addition to the additional longitudinal diffraction wave $\Phi - \varphi_a$, where both types ψ_1 and ψ_2, are distinctive by the direction of displacement vector polarization. The additional perturbations $\Phi - \varphi_a, \psi_1$ and ψ_2 describe the influence of the elasticity.

Setting $f\ (\tau) = r_0\eta\ (\tau)$ in (2. 14) and then letting $r_0 \to \infty$, it can be shown that $\psi_2 \to 0$, and Φ and ψ_1 yield the solution of the problem of diffraction of a plane longitudinal step wave by a wedge, in the limit. This solution agrees with the known solution [5] to notation accuracy (if slight inaccuracies are corrected for φ and ψ in (2. 9) from [5] : if the lost factor $1/_2$ is taken into account in the additional terms to the acoustic solution, and the following misprints are corrected ; replace one of the factors $\cos k\theta_0$ by $\cos k\theta$ in the expression for φ and the factor $(b / a)^k$ by $(b / a)^{-k}$ in the expression for ψ). We have

$$\Phi = \varphi_a{}^\circ + \frac{4 \sin l\pi \cos l\theta_0}{\pi\,(1 + \gamma^{2l})}\left[P\left(\frac{\tau}{r}\right) - 1 / P\left(\frac{\tau}{r}\right)\right]\eta\,(\tau - r) \tag{2.16}$$

$$\psi_1 = \frac{4\gamma^l \sin l\pi}{\pi\,(1 + \gamma^{2l})} \cos l\theta_0 \sin l\theta \left[P\left(\frac{\tau}{r\gamma}\right) - 1 / P\left(\frac{\tau}{r\gamma}\right)\right]\eta\,(\tau - r\gamma)$$

$$P\ (x) = [x + (x^2 - 1)^{1/2}]^l$$

Here φ_a^0 is the solution of the corresponding acoustic problem.

3. Let us examine the most interesting incident wave case in detail : (1.1) when $f(\tau) = -r_0\tau^2\eta(\tau)/2$. The stress on the front of such a compression wave undergoes a finite jump : $[\sigma_{nn}] = \sigma_{nn}{}^+ - \sigma_{nn}{}^- = -(\lambda + 2\mu)r_0(\tau + r_0)^{-1}$, where λ and μ are Lamé parameters, n is the normal to the wavefront, and the plus and minus signs, respectively, refer to domains behind and ahead of the wavefront (for $r_0 \to \infty$ this wave goes over into a plane wave with the potential

$$\varphi_0 = -[\tau + r\cos(\theta - \theta_0)]^2\eta[\tau + r\cos(\theta - \theta_0)]/2).$$

To investigate the solution it is sufficient to consider the case $0 < \theta_0 < \pi/l - \pi$ (Fig. 2a) in which all three are possible perturbed motion domains : a selected wave domain $(0 < \theta < \pi - \theta_0,\ [(\tau + r_0)^2 - z^2]^{1/2} - r_0 < r,\ \tau + r_0 > R(-\theta))$, a diffraction domain $(r < [(\tau + r_0)^2 - z^2]^{1/2} - r_0)$ and a shade domain $(\pi + \theta_0 < \theta < \pi/l)$.

The reflected wave potential

$$\varphi = -r_0[\tau + r_0 - R(-\theta)]^2\eta[\tau + r_0 - R(-\theta)]2^{-1}R^{-1}(-\theta)$$

is added to the incident wave potential upon passage through the reflected wavefront 2, and we have a finite jump in the normal stress on the reflected wavefront equal to $[\sigma_{nn}] = -r_0(\lambda + 2\mu)(\tau + r_0)^{-1}$ (because $\partial^2\Phi/\partial n^2$ undergoes a discontinuity).

The strains, and therefore the stresses , are discontinuous upon passage through the longitudinal diffraction wavefront 3, but the derivative of the strain with respect to the normal direction to the front $\partial\varepsilon_{nn}/\partial n$ undergoes a discontinuity of the second kind (because $\partial^3\Phi/\partial n^3$ undergoes such a discontinuity). This derivative is finite approaching the front from the domain ahead of the front but has a singularity on the order of $\varepsilon^{-1/2}$ for an approach from the domain behind the front. Hence, both the acoustic solution φ_a and the additional elastic member have singularities on the order of $\varepsilon^{-1/2}$.

The strains are continuous at the front of the transverse wave 4 but the derivatives of the strain components with respect to the normal $\partial\varepsilon_{\theta n}/\partial n$ and $\partial\varepsilon_{\nu n}/\partial n$ undergo discontinuities of the second kind (where ν is measured along the line of intersection of the transverse wavefront and the plane $\theta = \text{const}$) since the derivatives $\partial^3\psi_1/\partial n^3$ and $\partial^4\psi_2/\partial n^4$, respectively, undergo discontinuities at this front. These derivatives are finite for an approach from outside $(\tau + r_0 < R_1)$ and have a singularity of order $\varepsilon^{-1/2}$ for an approach from within $(\tau + r_0 > R_1)$.

It is seen from the investigation presented that the additional elastic part of the solution is commensurate in magnitude with the diffraction part of the acoustic solution not only in the neighborhood of the wedge edge but also near the diffraction wavefront 3, and therefore, the elastic problem differs substantially from the acoustic problem not only near the wedge edge, but in the whole diffraction domain $r + r_0 < [(\tau + r_0)^2 - z^2]^{1/2}$ generally.

In the symmetric case $(\theta_0 = \pi/(2l))$ curves of the stress distribution $(-\sigma_{\theta\theta})$ as functions of r/r_1 are presented in Fig. 3 (r_1 is the coordinate of the point A in Fig. 2b) along a side face of the wedge OA in the zone $z = 0$ plane for the apex angle $\beta = \pi/6$ and $\lambda/\mu = 2$.

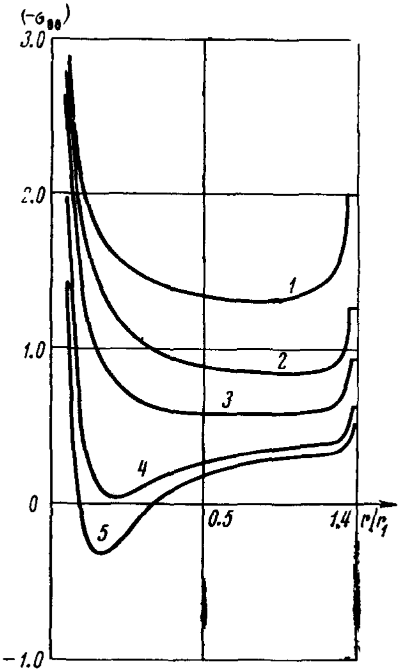

Fig. 3

Curve *1* is given for the case of plane compression wave incidence.

$$\varphi_0 = -[2\lambda + 4\mu \sin^2(\beta / 2)]^{-1} \times [\tau + r\cos(\theta - \theta_0)]^2 \eta\, [\tau + r\cos(\theta - \theta_0)] \quad (3.1)$$

Curves 2 - 5 are stress distributions on the wedge for $\tau / r_0 = 0.5,\ 1.0,\ 2.0,\ 2.5$, respectively, upon incidence of the spherical compression wave $\varphi_0 = -r_0 R^{-1} [2\lambda + 4\mu \sin^2(\beta/2)]^{-1} (\tau + r_0 - R)^2 \times \eta(\tau + r_0 - R)$ which goes over into the plane wave (3.1) as $r_0 \to \infty$. It is seen from a comparison between the results obtained for plane and spherical waves that the influence of sphericity of the incident wave front grows with the lapse of time and the stress on the section of the wedge side wall becomes tensile for $\tau \gtrsim 2r_0$. If the physical contact properties are hence such that tensile stresses cannot be transmitted then the phenomenon of "peeling off" occurs, and therefore, the solution obtained for the problem in this case (with the boundary conditions $v_\theta = 0,\ \sigma_{\theta r} = \sigma_{\theta z} = 0$) is suitable up to a definite time τ_1, when tensile stresses first appear on some part of the wedge surface. Other boundary conditions must be posed on these parts of the surface starting with this time.

As an analytical investigation and numerical computations show, tensile stresses appear for any wedge angle $\beta < \pi$ in the symmetric case, and the time of their appearance τ_1 grows without limit as $\beta \to \pi$ and the stresses are always compressive in the limit case $\beta = \pi$

Let us note that in the absence of symmetry $(\theta_0 \neq \pi / (2l))$ there are cases when the tensile stresses exist at any time (as, for example, during diffractions by a smooth solid plate for $\theta_0 = \pi / 2$, when the tensile stresses on the shaded side appear simultaneously with the formation of a perturbed motion domain).

It should here be mentioned that even when the physical contact properties do not permit transmission of the tensile stresses, the solution of the problem with tensile stresses can be given physical meaning if it is assumed that the elastic medium under consideration is already pre-compressed statically even before the beginning of the diffraction process, so that the resultant stresses on the contact turn out to be compressive.

In addition, let us note that the stress $\sigma_{\theta\theta}$ for both plane and spherical waves has a singularity on the order of r^{2l-2} upon approaching the wedge edge in the symmetric case.

4. A solution of the problem of diffraction of an elastic spherical wave by a wedge with apex angles $\beta \geqslant \pi$ can be obtained from the solution with angles

$\beta = 2\pi - \pi / l$ $(1/2 \leqslant l < 1)$, satisfying the condition $\beta < \pi$. In fact, the symmetric part of the solution (2.14), relative to the bisector plane of the wedge (we use the notation Φ^s and ψ_j^s, $j = 1, 2$) satisfies the following conditions on that plane: $\partial\Phi^s / \partial\theta = \psi_1^s = \partial\psi_2^s \partial\theta = 0$ and therefore, yields the solution of the problem of diffraction with the apex angle $\beta_1 = 2\pi - \pi / l_1$, where $l_1 = 2l$ $(1 \leqslant l_1 < 2)$. Here the angle β_1 satisfies the conditions $\pi \leqslant \beta_1 < 3\pi / 2$. Again extracting the symmetric part from the solution with β_1, we obtain the solution for $\beta_2 = 2\pi - \pi / l_2$ $(2 \leqslant l_2 = 2l_1 < 4)$, which satisfies the inequality $3\pi / 2 \leqslant \beta_2 < 7\pi / 4$ etc. After the n-th operation we obtain the solution for $\beta_n = 2\pi - \pi / l_n$ $(2^{n-1} \leqslant l_n = 2^n l < 2^n)$, satisfying the condition $2\pi - \pi / 2^{n-1} \leqslant \beta_n < 2\pi - \pi \cdot 2^n$. Therefore, the solution can be obtained for any apex angle β within the limits $0 \leqslant \beta < 2\pi$.

REFERENCES

1. Petrashen', G. I., Nikolaev, B. G. and Kouzov, D. P., On the method of series in the theory of wave diffraction from plane angular domains. Uchen. Zap. Leningrad Univ., № 246, 1958.
2. Borovikov, V. A., On the three-dimensional problem of diffraction by a prism, Dokl. Akad. Nauk SSSR, Vol. 148, № 3, 1963.
3. Filippov, A. F., Diffraction of an arbitrary acoustic wave by a wedge. PMM Vol. 28, № 2, 1964.
4. Thompson, J. H., Closed solutions for wedge diffractions. SIAM, J. Appl. Math., Vol. 22, № 2, 1972.
5. Kostrov, B. V., Diffraction of a plane wave by a rigid wedge in an unbounded elastic medium in absence of friction. PMM Vol. 30, № 1, 1966.
6. Sveklo, V. A. and Siukiiainen, V. A., Plane elastic wave diffraction with respect to a corner. Dokl. Akad. Nauk SSSR, Vol. 119, № 6, 1958.
7. Rakhmatulin, Kh. A., Saatov, Ia. U., Sabodash, P. F. and Filippov, I. G., Two-dimensional Problems on Unsteady Motion of Compressible Media. "Fan", Tashkent, 1969.
8. Guz', A. N. and Golovchan, V. T., Elastic Wave Diffraction in Multiconnected Bodies. "Naukova Dumka", Kiev, 1972.
9. Friedlander, F., Sound Pulses. Izd. Inostr. Lit., Moscow, 1962.
10. Bateman, H. and Erdelyi, A., Tables of Integral Transforms, New York, McGraw-Hill, 1954.
11. Ditkin, V. A. and Prudnikov, A. P., Integral Transforms and Operational Calculus. (English translation), Pergamon Press, Book № 10044, 1966.
12. Barr, G. E., On the diffraction of a cylindrical pulse by a half-plane. Quart. Appl. Math., Vol. 25, № 2, 1967.

Translated by M. D. F.